

Last Time: Diagonalization of matrices.

Algorithm: Let M be square matrix.

- ① Characteristic Poly $P_M(\lambda) = \det(M - \lambda I)$
- ② Solve $P_M(\lambda)$ for eigenvalues.
- ③ Build a basis for \mathbb{R}^n (or \mathbb{C}^n) of eigenvalues (Eigenbasis).
↳ Compute bases of each Eigenspace.
- ④ Supposing each eigenvalue λ has geom mult = alg mult,
the result of these computations is a basis E .
- ⑤ Realize $M = PDP^{-1}$ where $P = [E] = \text{Rep}_{E, E_n}(id)$,
and $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is the matrix of eigenvalues.

Ex: We diagonalize $M = \begin{bmatrix} -9 & -4 \\ 24 & 11 \end{bmatrix}$.

Char Poly: $P_M(\lambda) = \det(M - \lambda I) = \det \begin{bmatrix} -9-\lambda & -4 \\ 24 & 11-\lambda \end{bmatrix}$

$$\begin{aligned} &= (-9-\lambda)(11-\lambda) - 24(-4) \\ &= -99 - 2\lambda + \lambda^2 + 96 \\ &= \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) \\ &= (3 - \lambda)(-1 - \lambda) \end{aligned}$$

\therefore We have eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$

(NB: Have 2 distinct e-values for this 2×2 matrix,
so M is automatically diagonalizable \smile).

$\lambda_1 = 3$: $V_{\lambda_1} = \text{null}(M - \lambda_1 I) = \text{null} \begin{bmatrix} -9-3 & -4 \\ 24 & 11-3 \end{bmatrix} = \text{null} \begin{bmatrix} -12 & -4 \\ 24 & 8 \end{bmatrix}$

$$= \text{null} \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} = \text{null} \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_{\lambda_1}$ iff $3x + y = 0$ iff $y = -3x$

iff $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -3x \end{bmatrix} = x \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\therefore B_{\lambda_1} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$ is a basis of V_{λ_1} .

If M is $n \times n$
and M has
 n distinct
e-values, then
 M is diag'ble.

$$\underline{\lambda_2 = -1}: V_{\lambda_2} = \text{null}(M - \lambda_2 I) = \text{null} \begin{bmatrix} -9+1 & -4 \\ 24 & 11+1 \end{bmatrix} = \text{null} \begin{bmatrix} -8 & -4 \\ 24 & 12 \end{bmatrix} \\ = \text{null} \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \text{null} \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_{\lambda_2} \text{ iff } 2x + y = 0 \text{ iff } y = -2x \\ \text{iff } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -2x \end{bmatrix} = x \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

$$\therefore B_{\lambda_2} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} \text{ is a basis of } V_{\lambda_2}$$

Eigenbasis: Let $E = B_{\lambda_1} \cup B_{\lambda_2} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$. Now let

$$P = \text{Rep}_{E, E_2}(\text{id}) = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

Check: We'll verify $M = PDP^{-1}$: $\left(P^{-1} = \frac{1}{2-(-3)} \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix} \right)$

$$PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix} \\ = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -6 & -3 \\ -3 & -1 \end{bmatrix} \\ = \begin{bmatrix} -9 & -4 \\ 24 & 11 \end{bmatrix} = M \quad \text{✓}$$

Ex: Let $M = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$. Diagonalize if possible.

Sol: Char poly: $p_M(\lambda) = \det(M - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 + 1$

$$\therefore \text{eigenvalues: } p_M(\lambda) = 0 \Leftrightarrow (2-\lambda)^2 = -1 \Leftrightarrow 2-\lambda = \pm i$$

$$\Leftrightarrow \lambda = 2 \pm i$$

(NB: Not every diag'ble matrix is
diag'ble over \mathbb{R} ...)

$\therefore M$ diag'ble over \mathbb{C} .
↑ complex e-val.

$$\underline{\lambda_1 = 2+i}: V_{\lambda_1} = \text{null}(M - \lambda_1 I) = \text{null} \begin{bmatrix} 2-(2+i) & 1 \\ -1 & 2-(2+i) \end{bmatrix}$$

$$= \text{null} \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} = \text{null} \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_{\lambda_1} \text{ iff } x + iy = 0 \text{ iff } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -iy \\ y \end{bmatrix} = y \begin{bmatrix} -i \\ 1 \end{bmatrix}. \therefore B_{\lambda_1} = \left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}.$$

$$\lambda_2 = 2 - i: V_{\lambda_2} = \text{null}(M - \lambda_2 I) = \text{null} \begin{bmatrix} 2 - (2-i) & 1 \\ 2 - (2-i) & 1 \end{bmatrix} = \text{null} \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \\ = \text{null} \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_{\lambda_2} \text{ iff } x - iy = 0 \text{ iff } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} iy \\ y \end{bmatrix} = y \begin{bmatrix} i \\ 1 \end{bmatrix}. \therefore B_{\lambda_2} = \left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}.$$

Eigenbasis: $E = B_{\lambda_1} \cup B_{\lambda_2} = \left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix}, \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$ is a basis of \mathbb{C}^2 .

$$\therefore M = PDP^{-1} \text{ for } P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}.$$

Check: $PDP^{-1} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix} \cdot \frac{1}{2} i \begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix} \quad (P^{-1} = \frac{1}{-i-i} \begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix})$

$$= \frac{1}{2} i \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2+i & -2i+1 \\ -2+i & -2i-1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix} \begin{bmatrix} 2+i & 1-2i \\ -2+i & -1-2i \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2+i+2-i & 1-2i+1+2i \\ 2i-1-2i-1 & i+2-i+2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \underline{M} \quad \text{☺}$$

(NB: $M = PDP^{-1} \Leftrightarrow P^{-1}M = DP^{-1} \Leftrightarrow P^{-1}MP = D$) \square

Ex: Diagonalize $M = \begin{bmatrix} -4 & 1 \\ -1 & -6 \end{bmatrix}$ if possible.

Sol: $p_M(\lambda) = \det(M - \lambda I) = \det \begin{bmatrix} -4-\lambda & 1 \\ -1 & -6-\lambda \end{bmatrix} = (4+\lambda)(6+\lambda) + 1 \\ = (24 + 10\lambda + \lambda^2) + 1 = \lambda^2 + 10\lambda + 25 = (\lambda+5)^2 = \underline{(-5-\lambda)^2}$

$$\therefore \lambda_1 = -5: V_{\lambda_1} = \text{null} \begin{bmatrix} -4+5 & 1 \\ -1 & -6+5 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_{\lambda_1} \text{ iff } x+y=0 \text{ iff } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore B_{\lambda_1} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \text{ is a basis of } V_{\lambda_1}.$$

NB: Geometric mult of λ_1 is 1 < 2 and 2 is the alg mult of λ_1 , so M is not Diagonalizable over \mathbb{C} . \square

Comment: If M has exactly 1 e-value, it diagonalizes iff it was already diagonal.
 $PDP^{-1} = P(\lambda I)P^{-1} = \lambda(PIP^{-1}) = \lambda(PP^{-1}) = \lambda I = D$
 when D has a unique eigenvalue λ .

Ex: Diagonalize $M = \begin{bmatrix} -5 & 0 & 6 \\ -3 & 1 & 3 \\ -3 & 0 & 4 \end{bmatrix}$ if possible.

Sol: $p_n(\lambda) = \det(M - \lambda I) = \det \begin{bmatrix} -5-\lambda & 0 & 6 \\ -3 & 1-\lambda & 3 \\ -3 & 0 & 4-\lambda \end{bmatrix}$
 $= -0 + (1-\lambda) \det \begin{bmatrix} -5-\lambda & 6 \\ -3 & 4-\lambda \end{bmatrix} - 0$
 $= (1-\lambda) ((-5-\lambda)(4-\lambda) - (-3)6)$
 $= (1-\lambda) (-20 + \lambda + \lambda^2 + 18)$
 $= (1-\lambda) (\lambda^2 + \lambda - 2) = (1-\lambda) (\lambda - 1) (\lambda + 2)$
 $= (1-\lambda)^2 (-2-\lambda)$

\therefore have $\lambda_1 = 1$ an e-value of alg mult 2 and
 $\lambda_2 = -2$ an e-value of alg mult 1.

$\lambda_1 = 1$: $V_{\lambda_1} = \text{null}(M - \lambda_1 I) = \text{null} \begin{bmatrix} -5-1 & 0 & 6 \\ -3 & 1-1 & 3 \\ -3 & 0 & 4-1 \end{bmatrix} = \text{null} \begin{bmatrix} -6 & 0 & 6 \\ -3 & 0 & 3 \\ -3 & 0 & 3 \end{bmatrix}$
 $= \text{null} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V_{\lambda_1}$ iff $x - z = 0$ iff $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\therefore B_{\lambda_1} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis of V_{λ_1} .

(so λ_1 has geometric mult equal to its alg mult \smile).

$\lambda_2 = -2$: $V_{\lambda_2} = \text{null}(M - \lambda_2 I) = \text{null} \begin{bmatrix} -5+2 & 0 & 6 \\ -3 & 1+2 & 3 \\ -3 & 0 & 4+2 \end{bmatrix} = \text{null} \begin{bmatrix} -3 & 0 & 6 \\ -3 & 3 & 3 \\ -3 & 0 & 6 \end{bmatrix}$
 $= \text{null} \begin{bmatrix} 1 & 0 & -2 \\ -3 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$$\therefore V_{\lambda_2} \ni \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ iff } \begin{cases} x - 2z = 0 \\ y - z = 0 \end{cases} \text{ iff } \begin{cases} x = 2t \\ y = t \\ z = t \end{cases} \text{ iff } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore B_{\lambda_2} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ is a basis of } V_{\lambda_2}.$$

Eigenbasis: $E = B_{\lambda_1} \cup B_{\lambda_2} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis of \mathbb{R}^3 ,

so M diagonalizes over \mathbb{R} \square .

Indeed $M = P D P^{-1}$ for $P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

P^{-1} computation $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array} \right]$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array} \right] \rightsquigarrow \left[I \mid \begin{array}{ccc} -1 & 0 & 2 \\ -1 & 1 & 1 \\ -1 & 0 & -1 \end{array} \right]$$

$$\therefore P^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ -1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}.$$

Check: $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ -1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \stackrel{?}{=} M \quad \square$